Old UNM Algebra Qualifier Problems

Based on Chapters 3 of Dummit–Foote

Fifteen-year collection: January 2012–August 2023

Scope. This packet uses the third edition of Dummit and Foote, *Abstract Algebra*. The first section contains problems that can reasonably be attempted after Chapter 3. The second section records closely related problems whose standard solutions use some later material.

I. Core problems from Chapter 3

1. January 2013 Qualifier, Problem 2

Suppose that N is a normal subgroup of a group G, with $|N| < \infty$. Suppose also that H is a subgroup of G with $[G : H] < \infty$ and that $|N|$ and $[G : H]$ are relatively prime. Show that $N \leq H$.

2. January 2013 Qualifier, Problem 4

Let F be the free group on two generators and let $[F, F]$ be its commutator subgroup. Prove that

$$F/[F, F] \cong \mathbb{Z} \times \mathbb{Z}.$$

3. August 2013 Qualifier, Problem 1

Prove that if H and K are finite subgroups of G whose orders are relatively prime, then $H \cap K = \{e\}$.

4. August 2013 Qualifier, Problem 2

Let G be a group and let H be a subgroup contained in the center of G. Show that if G/H is cyclic, then G is abelian.

5. January 2015 Qualifier, Problem 9

Prove that the group S_4 is solvable.

6. August 2016 Qualifier, Problem 3

Prove that the group of invertible upper-triangular $n \times n$ complex matrices is solvable.

Note: This problem is repeated in **August 2018 Qualifier, Problem 7** and in **August 2023 Qualifier, Problem 3**.

7. August 2017 Qualifier, Problem 1

Suppose N and H are normal subgroups of G such that G/N and G/H are abelian. Show that $G/(H \cap N)$ is abelian.

8. August 2018 Qualifier, Problem 2

Let N be a normal subgroup of a group G. Show that N contains the commutator subgroup if and only if G/N is abelian.

9. January 2020 Qualifier, Problem 2

Suppose H and N are subgroups of a group G, with $N \trianglelefteq G$, $|H| = m < \infty$, and $[G : N] = n < \infty$. If m and n are relatively prime, prove that $H \leq N$.

10. January 2021 Qualifier, Problem 2

Prove that the group of invertible 2×2 complex matrices is not solvable.

11. January 2023 Qualifier, Problem 1

Suppose N is a normal subgroup of a finite group G and

$$\gcd\big(|N|, (G : N)\big) = 1.$$

Show that N is the unique subgroup of G of order $|N|$.

12. August 2024 Qualifier, Problem 1

Suppose

$$1 \longrightarrow H \xrightarrow{\alpha} G \xrightarrow{\beta} K \longrightarrow 1$$

is an exact sequence of groups such that H and K are abelian and $\alpha(H)$ is central in G. Let a be a fixed element of G. Show that

$$\varphi : G \longrightarrow G, \qquad \varphi(g) = aga^{-1}g^{-1},$$

is a group homomorphism.

II. Closely related or later-material problems

1. January 2012 Qualifier, Problem 2

Prove that every group of order pq, where p and q are unequal primes, is solvable.

2. August 2012 Qualifier, Problem 1

Let p, q, r be primes. Show that no group of order pqr is simple.

3. January 2013 Qualifier, Problem 3

Show that there are no simple groups of order 196.

4. August 2013 Qualifier, Problem 3

Show that no group of order 132 is simple.

5. January 2014 Qualifier, Problem 3

Let G be a group of order 105 with a normal Sylow 3-subgroup. Show that G is abelian.

6. January 2015 Qualifier, Problem 1

Let G be a nonabelian group of order p^3. Show that

$$Z(G) = G' = [G, G].$$

7. January 2015 Qualifier, Problem 2

Show that there is no simple group of order 192.

8. August 2017 Qualifier, Problem 2

Show that every group of order 100 is not simple.

9. January 2018 Qualifier, Problem 6

Let p be a prime. Show that for every nonabelian group G of order p^3, the commutator subgroup of G is equal to the center of G.

10. August 2018 Qualifier, Problem 1

Show that every group of order $3^3 \cdot 13$ is not simple.

11. January 2019 Qualifier, Problem 2

Show that no group of order 52 is simple.

12. August 2019 Qualifier, Problem 3

Let K be a field and let G be the set of all functions $f : K \to K$ of the form

$$f(x) = ax + b,$$

where $a, b \in K$ and $a \neq 0$. View G as a group under composition. Prove that G is solvable. Is G nilpotent? Justify your answer.

13. January 2020 Qualifier, Problem 1

Let G be a group of order 402. Prove that G is not simple.

14. August 2021 Qualifier, Problem 1

Show that no group of order 72 is simple.

15. January 2022 Qualifier, Problem 1

Show that no group of order 132 is simple.

16. January 2022 Qualifier, Problem 2

Let p be an odd prime and let G be a group of order p^3. Show that

$$Z(G) = [G, G].$$

Correction to the printed problem. As written, the assertion is false: for example, if $G = C_{p^3}$, then $Z(G) = G$ while $[G, G] = 1$. The intended statement requires G to be nonabelian.

17. January 2024 Qualifier, Problem 1

Show that there is no simple group of order 80.

18. August 2024 Qualifier, Problem 2

Let G be a group of order $3 \cdot 2^n$, where $n \geq 1$. Show that G is solvable.

Source

University of New Mexico, Department of Mathematics and Statistics, Past Qualifying Exams – Algebra. Individual exam dates and problem numbers are recorded above.